Assignment 4

Coverage: 15.5 in Text. (Will give more examples on the coming Monday)

Exercises: 15.5. no 3, 4, 21, 23, 25, 28, 29, 32, 33, 38, 39, 42, 44.

Submit 15.5 no. 29, 42 and 44 by Oct 5.

On the Notion of Area

I add a few words on the notion of area. In fact, it is a significant step in mathematics to define area rigorously using integration. This is for optional reading, but I recommend you to read and think about it.

Since high school we have been familiar with the notion of area. Formulas for the area of rectangles, disks, triangles as well as other common geometric figures are well-known. For a given region, although we do not know how to compute its area exactly, we know its area exists. However, we do not have a mathematical definition of area.

In this chapter, we have learned how to use Riemann integral to give a mathematical definition to the notion of area. For a set E in the plane, we define its area to be

$$|E| \equiv \iint_E \chi_E \, dA \ ,$$

provided χ_E is integrable. When E is a region, its characteristic function χ_E is integrable and hence the area of E is well-defined.

After reaching a mathematical definition, our immediate job is to verify that for common geometric figures, the newly defined concept is the same as the old one.

Our verification starts with the rectangles. Let $R = [a,b] \times [c,d]$ and P a partition on it. Its Riemann sum is

$$R(\chi_R, P) = \sum_{i,j} 1 \, \Delta x_i \Delta y_j \ .$$

We have

$$\begin{aligned} R(\chi_R, P) &= \sum_j \sum_i \Delta x_i \Delta y_j \\ &= \sum_j \Delta y_j \sum_i \Delta x_i \\ &= \sum_j \Delta y_j (b-a) \\ &= (d-c)(b-a) \end{aligned}$$

which is a constant. Hence

$$|R| = (b-a)(d-c)$$

which is equal to the old formula that the area is the product of the width and the height.

For the disk $x^2 + y^2 = R^2$ we use Fubini's theorem to get

$$\begin{aligned} |D_R| &= \iint_{D_R} \chi_{D_R}\, dA \\ &= \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dydx \\ &= 2\int_{-R}^{R} \sqrt{R^2-x^2}\, dx \\ &= \pi R^2 \ . \end{aligned}$$

Again this is consistent with the old formula.

The area of a sector of open angle θ is $\theta R^2/2$. We have used this formula in the derivation of the formula for polar coordinates, see Theorem 1.11. It is a good exercise to derive this formula using our new definition. I did it in class. Try to reproduce it by yourself.

One can also verify that our new definition of area is invariant under Euclidean motions. A Euclidean motion consists of a finite combination of translations and rotations (Wiki for more information). It suffices to verify the invariance under a single translation and rotation. Furthermore, it suffices to verify it for rectangles (since the Riemann sums, which approximates the integral, are expressed in terms of the area of subrectangles.) For a translation this is easy. For a rotation, one can proceed by brute force using Fubini's theorem. Alternatively, we could appeal to the change of variables formula in the next chapter.